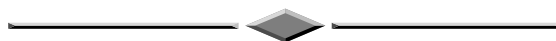


# Maintaining Consistent Global Asset Views (with a Little Help from Euclid)

Brian D. Singer, Kevin Terhaar, and John Zerolis

*Global economic and capital market integration has led to a host of new concerns for plan sponsors and money managers. In such an environment, ensuring consistent capital market expectations, policies, and strategies among different base currencies is important. A primary difficulty is translating risk and correlation estimates from the home currency into any other currency. We provide a method for converting risk and correlation views. The method makes use of a geometric interpretation of risks and correlations, dramatically simplifying the requisite calculations.*



Risk analysis and management are commanding ever-greater resources at financial and nonfinancial firms. Well-publicized events in the marketplace have spurred these institutions, and subsequently their regulatory overseers, into action. Money managers, however, have been somewhat slow to embrace recent advances in risk analysis and management. Money management saw the largest advance in risk analysis when Harry Markowitz published his seminal article "Portfolio Selection" in 1952. Since then, many of the developments have gone unrecognized or unused.

Money management is becoming more and more globalized. Portfolios that once comprised only U.S. assets increasingly include some normal allocation to non-U.S. assets. Today, U.S. pension allocations to non-U.S. assets are at 11 percent, up from 4 percent in 1991. Now, global asset management means much more than simply managing U.S.-based assets in a global arena. It also means managing assets for clients that are domiciled around the globe.

We applied a recent quantitative insight to a money-management problem that arises from globalization: how to translate return and volatility views from the perspective of one's home currency into any other base currency in a consistent and efficient manner.

To accomplish this objective, we used geometry to characterize currency risks and correlations.

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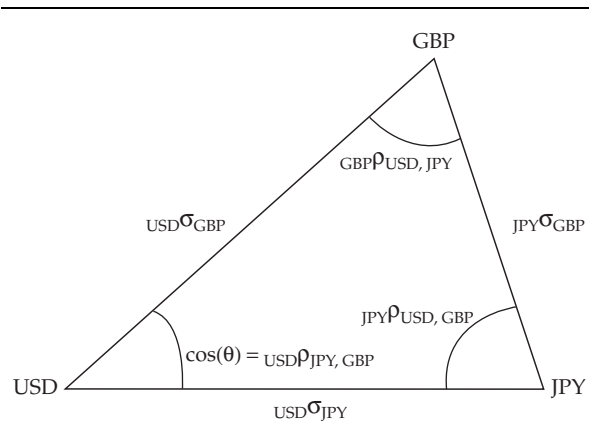
*Brian D. Singer, CFA, is a senior asset allocation analyst and Kevin Terhaar, CFA, is a senior analyst at Brinson Partners. John Zerolis is Director of Risk Control at the Swiss Bank Corporation and a lecturer in the Financial Mathematics Program at the University of Chicago.*

This geometric perspective can be used to convert risk and correlation forecasts from one base currency into another, thus allowing us to maintain consistent currency views and strategies regardless of the base currency. The key to the geometric approach involves constructing triangles with vertices representing the alternate base currencies and the sides corresponding to risks. An example of such a triangle is shown in Figure 1. We will show that the cosines of the angles of such a triangle are the correlations between the currencies.

Within this framework, trigonometric calculations allow us to translate currency risk and correlation views from one base currency to another. The schematic of Figure 1 is relevant because it creates consistency. For example, given that the investor is U.S. dollar (USD) based and given a statement about pound sterling (GBP) and yen (JPY) volatilities and their correlation, we can construct the entire triangle and derive *all* other volatilities and correlations. Translating historical volatilities and correlations into other base currencies is trivial; the historical returns need only be expressed in the desired base currency and all volatilities and correlations recomputed.

Using volatilities and correlations that are not historically derived poses problems for translation from one base currency to another. For example, option markets provide some useful forward-looking estimates of volatilities and even correlations. In many cases, however, the data on option prices are limited. Alternatively, univariate and multivariate GARCH (generalized autoregressive conditional heteroscedasticity) and exponential-weighting techniques are becoming increasingly popular in describing the tendency of volatilities and correlations to vary and cluster over time.

**Figure 1. Geometric Approach to Currency Translation**



Note: The subscripts before the Greek letters denote the base currency; subscripts after the letters denote the currencies being "viewed" from the base.

When historical data are the basis for forward-looking estimates, however, consideration of alternate views raises problems. Consider the problems involved in using historical data to analyze European currencies. If a U.S. dollar-based manager holds the opinion that European Monetary Union (EMU) is likely to transpire in 1999, then forward-looking risk analysis should incorporate higher correlations among European currencies than the historical data would imply.<sup>1</sup>

We have developed a geometric framework for efficiently converting a covariance matrix from one currency perspective to another, enabling money managers to maintain consistent risk and correlation views for investors of any domicile. Money managers can focus all their efforts on developing a covariance matrix from the perspective of one base currency. As new clients in different base currencies come under management, additional resources need not be spent developing new matrixes. Rather, one core, consistent matrix can be maintained and translated as necessary.

## Geometric Alternative

Modern statistical analysis has existed for a little

longer than a hundred years. Euclidean geometry, however, has been around for more than 2,000 years. Our knowledge of geometry, benefiting from this long history, is broad and deep. Moreover, as infants, we learn how to distinguish shapes and sizes in order to place round pegs in round holes and square pegs in square holes. Interpreting risk from the perspective of geometry enables us to draw on not only this extensive knowledge but also most individuals' natural visual intuition.<sup>2</sup>

Some simple examples using historical exchange rate changes from December 31, 1979, to December 31, 1996, illustrate the geometric interpretation of risk. The returns, risks, and correlations, as seen from a U.S.-dollar-based investor's perspective, are shown in Table 1. Because the value of the U.S. dollar does not vary, its return and volatility are zero; furthermore, all correlations with the U.S. dollar are zero.

According to Table 1, the Canadian dollar (CAD) has had lower volatility than the German mark (DEM) and the British pound. The correlation between the Canadian dollar and either of the European currencies also has been lower than the correlation between the pound and the mark.

Assume that a U.S. dollar-based investor hires a manager to manage a currency strategy against an equally weighted benchmark and that the data in Table 1 represent the manager's expectations regarding the future returns, risks, and correlations. The benchmark weights in Column A of Table 2 yield a portfolio that has an expected risk of 5.92 percent and an expected return of 0.05 percent. Column B is an optimization to exploit return and risk expectations by constructing a currency strategy with the highest expected return at the same level of risk as the benchmark.<sup>3</sup> This optimized currency strategy would be expected to provide a return of 0.54 percent, given weights of 64 percent in the Canadian dollar and 36 percent in the pound. The mark would not be held because its risk is high and its expected return is negative. The U.S. dollar, with an expected return of zero, would be held only to reduce risk.

What if the manager were to solve this problem

**Table 1. Currency Returns, Risks, and Correlations from a U.S. Dollar Perspective, December 31, 1979, through December 31, 1996**

Currency	Return	Standard Deviation	Correlations			
			USD	DEM	CAD	GBP
USD	0.0%	0.0%	1.00	0.00	0.00	0.00
DEM	-0.9	12.0	0.00	1.00	0.14	0.71
CAD	0.5	4.8	0.00	0.14	1.00	0.25
GBP	0.6	12.0	0.00	0.71	0.25	1.00

Note: Based on monthly logarithmic returns.

**Table 2. Benchmark and Portfolio Weights, Returns, and Risks, December 31, 1979, through December 31, 1996**

Currency	USD Perspective		DEM Perspective	
	Benchmark (A)	Portfolio (B)	Benchmark (C)	Portfolio (D)
USD	25%	0%	25%	0%
DEM	25	0	25	16
CAD	25	64	25	18
GBP	25	36	25	66
Return	0.05	0.54	0.95	1.24
Risk	5.92	5.92	7.20	7.20

Note: Based on monthly logarithmic returns.

for a mark-based investor? What if the manager had the same benchmark and the same mandate to achieve the highest return at the risk level of the benchmark? The returns, risks, and correlations would not be the same from a mark perspective. In Table 3, all exchange rate data are expressed in mark terms and the exchange rate returns, risks, and correlations are recomputed from a mark-based investor's viewpoint. The first point to note is that the risks changed considerably. For a German mark investor, the Canadian dollar is the highest-risk currency and the pound is the least risky of the foreign currencies. In addition, the correlation between the U.S. and Canadian dollars is very high, because they move together against the mark. Obviously, the currency strategy that was appropriate for the U.S. investor is likely to be far from optimal for the German investor.

Again, assuming that the historical data represent the manager's expectations, an equally weighted benchmark, shown in Column C of Table 2, would be expected to have a volatility, from a mark perspective, of 7.20 percent and a return of 0.95 percent. The optimal currency strategy, in Column D, has much less weight placed on the Canadian dollar and much more on the pound than in the case of the U.S. dollar-based investor. At the benchmark risk level of 7.20 percent, the optimal weights are 66 percent in the pound, 16 percent in the mark, and only 18 percent in the Canadian dollar. This strategy would be expected to provide a return of 1.24 percent, an advantage of almost 30

basis points more than the benchmark.

This approach is fine for those using historical data exclusively. The currency perspective is easily changed, and the new risks and correlations can be computed without much difficulty. But what happens if the correlations and risks are not derived historically? For example, a manager may hypothesize that in the future the United Kingdom will peg the pound to the euro (after EMU starts). This opinion might lead the manager to believe that the correlation between the German mark and the pound will likely be higher than the 0.71 in Table 1. Assume, for example, that the manager believes that correlation will increase to 0.95. Using the higher GBP/DEM correlation, the manager finds that an equally weighted U.S. dollar-based benchmark (Column A of Table 4) has an expected return of 0.05 percent and a standard deviation of 6.27 percent. An optimized portfolio (Table 4, Column B) at the same risk level as the benchmark would be allocated 59 percent to the Canadian dollar and 41 percent to the pound. The portfolio has a 0.54 percent expected return. The DEM/GBP correlation change had essentially no impact on the U.S. client's optimal portfolio. The modest shift from Canadian dollars into pounds reflects the increase in benchmark and, therefore, portfolio risk from 5.92 percent (Table 2) to 6.27 percent (Table 4) that results from the higher correlation.

**The Geometric Solution.** The historical data showed that the risks and correlations of currencies

**Table 3. Currency Returns, Risks, and Correlations from a German Mark Perspective, December 31, 1979, through December 31, 1996**

Currency	Return	Standard Deviation	Correlations			
			USD	DEM	CAD	GBP
USD	0.9%	12.0%	1.00	0.00	0.92	0.38
DEM	0.0	0.0	0.00	1.00	0.00	0.00
CAD	1.4	12.3	0.92	0.00	1.00	0.43
GBP	1.5	9.1	0.38	0.00	0.43	1.00

Note: Based on monthly logarithmic returns.

**Table 4. Benchmark and Portfolio Weights, Returns, and Risks: Modified Data**

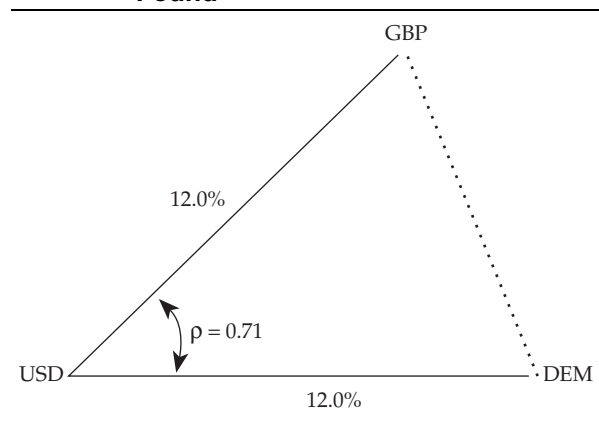
Currency	USD Perspective		DEM Perspective	
	Benchmark (A)	Portfolio (B)	Benchmark (C)	Portfolio (D)
USD	25%	0%	25%	0%
DEM	25	0	25	0
CAD	25	59	25	0
GBP	25	41	25	100
Return	0.05	0.54	0.95	1.50
Risk	6.27	6.27	6.24	3.80

can change dramatically, depending on which currency they are based in. Our expectation is that the increased correlation between the pound and the mark will imply, for a mark-based investor, a lower pound risk because it will be more closely tied to the domestic currency—much as the Canadian dollar is “low risk” to a U.S. investor. To determine the corresponding risks and correlations from a German mark perspective, we constructed a geometric representation of the problem.

The historical relationship among the U.S. dollar, the mark, and the pound can be represented by the triangle in Figure 2. From the U.S. dollar perspective—represented by the U.S. dollar vertex—the length of the line to each currency is given by its risk. So, based on the data in Table 1, the triangle is constructed with both the U.S. dollar–pound side and the U.S. dollar–mark side equal to 12.0 percent. Also from Table 1, the correlation between the mark and pound is 0.71. The dashed line merely connects the mark and pound vertexes; its length corresponds to the risk of the mark from a pound perspective or, equivalently, the pound risk from a mark perspective.

The standard deviation of the pound from a mark perspective is 9.1 percent (from Table 3). Figure 3 shows the general case of two currencies, X

**Figure 2. Relationship among the U.S. Dollar, the German Mark, and the British Pound**



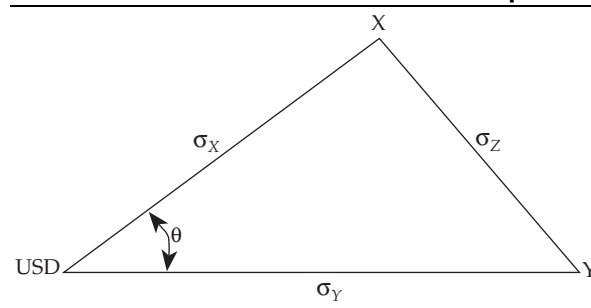
and Y, from a U.S. dollar perspective.

Appendix 1 provides a proof of the cosine rule, which states that

$$\cos(\theta) = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_z^2}{2\sigma_x\sigma_y} \tag{1}$$

The right-hand side of this relationship can be transformed into the equation for correlation. The first step is to substitute for the variances  $\sigma_x^2$ ,  $\sigma_y^2$ , and  $\sigma_z^2$ .

**Figure 3. Standard Deviations of Currencies X and Y from a U.S. Dollar Perspective**



Some intuition can help express the variance of  $z$  in terms of  $x$  and  $y$ . Assume that  $x_i$  is the continuously compounded percentage change in the pound from a U.S. dollar perspective and is equal to 0.5 percent. Also, assume that  $y_i$  is the percentage change in the mark, equal to 0.2 percent. The return on the pound from a mark perspective ( $z_i$ ) is equal to the pound return in U.S. dollars less the mark return in U.S. dollars. If, relative to the U.S. dollar, the pound appreciates by 0.5 percent and the mark appreciates by 0.2 percent, then the pound must appreciate relative to the mark by 0.5 percent minus 0.2 percent, or 0.3 percent. Thus,  $z_i$  merely equals  $x_i - y_i$ , and its variance is equal to

$$\frac{\sum [(x_i - y_i) - (\bar{x} - \bar{y})]^2}{n}$$

Let  $a = x_i - \bar{x}$  and  $b = y_i - \bar{y}$ . Then  $\sigma_x^2 = \frac{1}{n} \sum a^2$  and  $\sigma_y^2 = \frac{1}{n} \sum b^2$ . The variance of  $z$  can also be expressed as  $\sigma_z^2 = \frac{1}{n} \sum (a - b)^2$ .

Substituting into Equation 1 and partially rear-

ranging terms,

$$\cos(\theta) = \frac{\frac{1}{n} \sum [a^2 + b^2 - (a^2 + b^2 - 2ab)]}{2\sigma_x \sigma_y}$$

The  $a^2$ s and the  $b^2$ s in the numerator cancel so that

$$\cos(\theta) = \frac{\frac{2}{n} \sum ab}{2\sigma_x \sigma_y}$$

Substituting for  $a$  and  $b$  and rearranging terms provides

$$\cos(\theta) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n\sigma_x \sigma_y}$$

This last expression is the definition of the correlation between  $x$  and  $y$ , as seen from a U.S. dollar perspective. The vertices  $x$  and  $y$  can represent innumerable variables—the U.S. equity and bond markets, two currencies, a security and its benchmark—in other words, any two items whose correlation might be of interest.

Using this relationship between correlation and the cosine, the correlation between the German mark and the pound is

$$\begin{aligned} \cos(\theta) &= \text{USD}\rho_{\text{DEM,GBP}} \\ &= \frac{\text{USD}\sigma_{\text{GBP}}^2 + \text{USD}\sigma_{\text{DEM}}^2 - \text{DEM}\sigma_{\text{GBP}}^2}{2(\text{USD}\sigma_{\text{GBP}})(\text{USD}\sigma_{\text{DEM}})} \end{aligned}$$

Solving for  $\text{DEM}\sigma_{\text{GBP}}$ ,

$$\begin{aligned} \text{DEM}\sigma_{\text{GBP}}^2 &= \text{USD}\sigma_{\text{GBP}}^2 + \text{USD}\sigma_{\text{DEM}}^2 \\ &\quad - (\text{USD}\rho_{\text{DEM,GBP}})(2)(\text{USD}\sigma_{\text{GBP}})(\text{USD}\sigma_{\text{DEM}}) \\ &= (0.12)^2 + (0.12)^2 - 0.71(2)(0.12)(0.12) \\ \text{DEM}\sigma_{\text{GBP}} &= 9.1\% \end{aligned}$$

Use of the geometric representation has produced a risk that matches the historically derived pound risk in Table 3—shown as the mark–pound side in Figure 2. So, the transformation of currency risks to other base currencies can be accomplished in a relatively straightforward manner.

Knowing the standard deviation of the pound from a mark perspective enables us to use Equation 1 to compute the correlation between the pound and the U.S. dollar from a mark perspective:

$$\text{DEM}\rho_{\text{USD,GBP}} = \frac{\text{DEM}\sigma_{\text{USD}}^2 + \text{DEM}\sigma_{\text{GBP}}^2 - \text{USD}\sigma_{\text{GBP}}^2}{2(\text{DEM}\sigma_{\text{USD}})(\text{DEM}\sigma_{\text{GBP}})}$$

Using data from Table 1 and the value of  $\text{DEM}\sigma_{\text{GBP}}$  computed above,

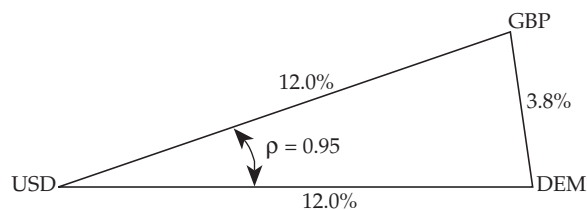
$$\begin{aligned} \text{DEM}\rho_{\text{USD,GBP}} &= \frac{(0.12)^2 + (0.091)^2 - (0.12)^2}{2(0.12)(0.091)} \\ &= 0.38. \end{aligned}$$

Similarly, the correlation between the U.S. dollar and the mark from a pound perspective is provided by the formula

$$\begin{aligned} \text{GBP}\rho_{\text{USD,DEM}} &= \frac{\text{GBP}\sigma_{\text{USD}}^2 + \text{GBP}\sigma_{\text{DEM}}^2 - \text{USD}\sigma_{\text{DEM}}^2}{2(\text{GBP}\sigma_{\text{USD}})(\text{GBP}\sigma_{\text{DEM}})} \\ &= \frac{(0.12)^2 + (0.091)^2 - (0.12)^2}{2(0.12)(0.091)} \\ &= 0.38. \end{aligned}$$

What happens when the correlation between the mark and the pound is increased from 0.71 to 0.95? For the U.S. client, the manager would use Table 1 with the altered DEM/GBP correlation. For the German client, what happens to the shape of the triangle in Figure 2? Intuitively, the risk of the pound to the German investor would decline: Because the pound would not be expected to fluctuate much against the mark, the mark–pound side of the triangle should become shorter. Given the same 12.0 percent volatility of the mark and the pound to the U.S. investor, those sides will not change. Therefore, the triangle would become “flatter,” as shown in Figure 4.

**Figure 4. Result of Increasing Correlation between the German Mark and the British Pound**



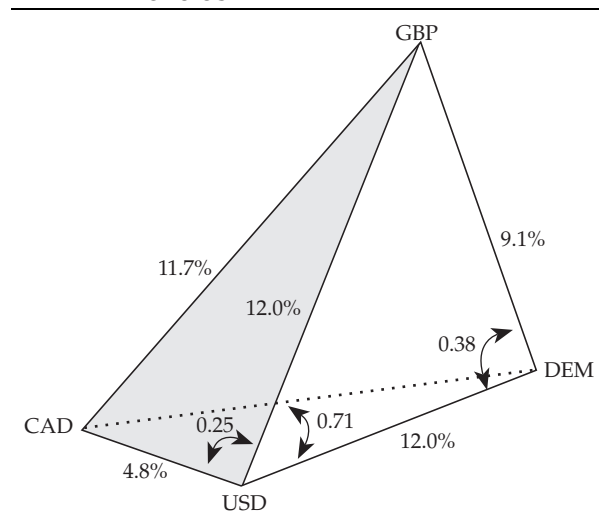
The standard deviation of the pound from a German mark perspective has clearly declined. The reader can confirm that the expected standard deviation would be 3.8 percent. Similarly, the reader can compute the correlations between the U.S. dollar and the pound from a mark perspective and between the U.S. dollar and the mark from a pound perspective.

**Many Currencies.** The four-currency case requires moving from the two-dimensional representation of a triangle to the three-dimensional tetrahedron. For example from a U.S. dollar perspective,

we move from a triangle with three vertexes to a tetrahedron with four vertexes, one for each of the four currencies. When converting to the mark perspective, we also need to find the new risks and correlations associated with the Canadian dollar.

Figure 5 provides the three-dimensional geometric analogy to the historical risks and correlations from Tables 1 and 3.<sup>5</sup> The front face of the tetrahedron is the triangle in Figure 2. The tetrahedron shows that the pound is less risky from the mark perspective than it is from a U.S. dollar or Canadian dollar perspective; the side connecting the pound and mark vertexes is shorter than the sides connecting the pound vertex to either the U.S. dollar or Canadian dollar vertex. Because of the high correlation from a mark view, the angle between the U.S. dollar and Canadian dollar sides from the mark vertex is very small.

**Figure 5. Geometric Approach with Four Currencies**



With four currencies, the risks can still be derived according to Equation 1 by simply triangulating any three currencies to compute the standard deviations from a non-U.S. dollar vertex. Deriving the correlation between two nonbase currencies from the perspective of the mark, however, requires an additional relationship (see Appendix 2).<sup>6</sup>

From Appendix 2, the correlation between the Canadian dollar and the pound from a mark perspective is given by

$$\text{DEM}\rho_{\text{CAD,GBP}} = \frac{\text{DEM}\sigma_{\text{CAD}}^2 + \text{DEM}\sigma_{\text{GBP}}^2 - [\text{USD}\sigma_{\text{CAD}}^2 + \text{USD}\sigma_{\text{GBP}}^2 - 2(\text{DEM}\sigma_{\text{GBP}})(\text{DEM}\sigma_{\text{CAD}})]}{2(\text{DEM}\sigma_{\text{GBP}})(\text{DEM}\sigma_{\text{CAD}})}$$

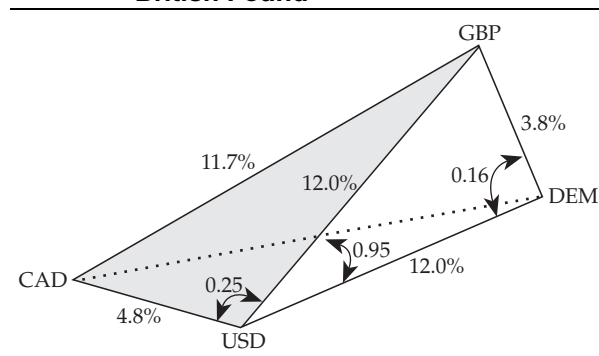
$$\begin{aligned} & \frac{2(\text{USD}\sigma_{\text{CAD}})(\text{USD}\sigma_{\text{GBP}})(\text{USD}\rho_{\text{CAD,GBP}})]}{2(\text{DEM}\sigma_{\text{GBP}})(\text{DEM}\sigma_{\text{CAD}})} \\ &= \frac{0.123^2 + 0.091^2 - [0.048^2 + 0.120^2 - 2(0.091)(0.123)]}{2(0.091)(0.123)} \\ &= \frac{2(0.048)(0.120)(0.25)}{2(0.091)(0.123)} \\ &= \frac{0.009586}{0.022386} \\ &= 0.43. \end{aligned} \tag{2}$$

This correlation corresponds to the actual historical correlation found in Table 3.

Now, we have all of the tools necessary to use the geometric framework to determine the implications for all four currencies of the proposed increase in the DEM/GBP correlation. Substitution of 0.95 for 0.71 in Equation 1 reduces the risk of the pound to a mark-based investor from 9.1 percent to 3.8 percent, which agrees with the intuition behind the shorter mark–pound side in the triangle in Figure 4. Increasing the DEM/GBP correlation takes the correlation between the U.S. dollar and pound, from a mark perspective, from 0.38 to 0.16. The angle at the mark vertex has become larger; thus the cosine and correlation have declined.<sup>7</sup> Intuitively, a pound that is more closely tied to the mark would show less correlation with the U.S. dollar.

When the correlation between the pound and the mark is increased (i.e., pound risk is reduced for a mark-based investor), the mark–pound side of the tetrahedron becomes shorter. Figure 6 shows the geometric representation based on the higher

**Figure 6. Geometry of Increasing Correlation between the German Mark and the British Pound**



DEM/GBP correlation. The front face of the tetrahedron is the triangle shown in Figure 4.

We have computed the risk of the pound and the correlation between the U.S. dollar and the

**Table 5. Currency Returns, Risks, and Correlations: Modified Correlation Matrix from a German Mark Perspective**

Currency	Return	Standard Deviation	Correlations			
			USD	DEM	CAD	GBP
USD	0.9%	12.0%	1.00	0.00	0.92	0.16
DEM	0.0	0.0	0.00	1.00	0.00	0.00
CAD	1.4	12.3	0.92	0.00	1.00	0.29
GBP	1.5	3.8	0.16	0.00	0.29	1.00

pound from a mark perspective based on the modified correlation matrix. To complete this matrix, presented in Table 5, the risk of the Canadian dollar and the correlations between the Canadian dollar and the U.S. dollar and the pound need to be computed. The reader can confirm these computations using Equations 1 and 2.

Table 2 presented the risks and returns for the equal-weighted benchmark and optimal currency portfolios for a mark-based investor based on historical data. Now, we can produce a similar table with the modified pound assumption. Columns C and D of Table 4 show, from a mark perspective, the benchmark and optimal portfolio returns and risks based on Table 5. The benchmark risk from Table 4 is 6.24 percent, almost 100 basis points less than the 7.20 percent risk indicated by the original covariance matrix. In this case, however, there is no way to generate an optimal currency strategy at that risk level. The British pound is simply too attractive, with low risk and the highest return; any attempt to hold the riskier U.S. and Canadian dollars results in inferior currency portfolios. The optimal portfolio at the highest feasible risk level holds only British pounds and has a volatility of 3.8 percent. German marks would be held only at lower risk levels.

Importantly, the portfolios shown in Table 4 are based on consistent currency return, risk, and correlation views. Despite altering the GBP/DEM correlation from a U.S. dollar perspective, we are able to compute a consistent set of optimization inputs from a German mark perspective.

## Conclusion

By using a geometric interpretation of risks and correlations, we can draw on a broad range of theories that simplify many aspects of modern finance. In this article, we made use of geometry in a unique application to the conversion of risk and correlation views from one currency into another. This approach provides an intuitive means of dealing with complex currency relationships. In fact, the conversion of a covariance matrix from one base currency into another can easily be computed with any spreadsheet package or with a few simple lines of computer code.<sup>8</sup> This approach obviates the need for deriving multiple covariance matrixes, one for each base currency, and enables a focus on developing an accurate home-currency matrix, which can then be translated into any base currency.

Consistent risk and correlation assumptions across base currencies are essential steps in the formulation of investment policies and strategies for clients based in different currencies. This currency translation methodology dictates consistent assumptions and enables managers to apply policies and strategies that are in concert for multiple base currencies.

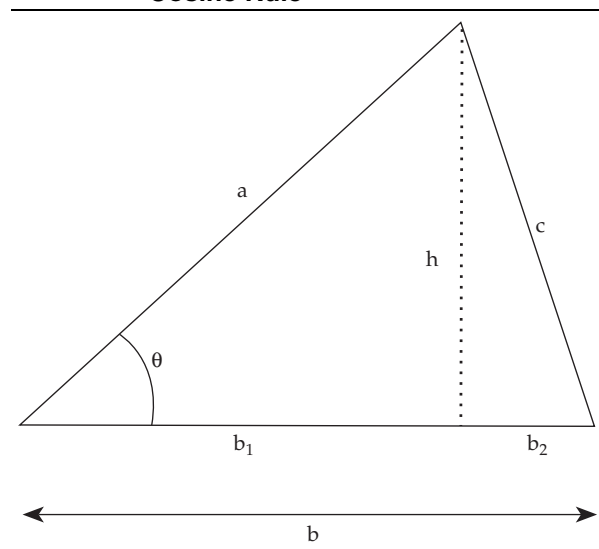
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## Appendix 1: The Cosine Rule

Consider the cosine rule (depicted in Figure A1):

**Figure A1. Geometric Representation of the Cosine Rule**



$$\cos(\theta) = \frac{a^2 + b^2 - c^2}{2ab}. \quad (\text{A1})$$

We would like to prove that

$$\frac{a^2 + b^2 - c^2}{2ab} = \frac{b_1}{a} = \cos(\theta).$$

Using the Pythagorean theorem, we know that  $c^2 = h^2 + b_2^2$  and  $a^2 = h^2 + b_1^2$ . Therefore, by substitution,  $c^2 = a^2 - b_1^2 + b_2^2$ .

By substituting the above expression for  $c^2$  into the numerator of Equation A1, we have

$$\begin{aligned} a^2 + b^2 - (a^2 - b_1^2 + b_2^2) &= b^2 + b_1^2 - b_2^2 \\ &= (b_1 + b_2)^2 + b_1^2 - b_2^2 \\ &= 2b_1^2 + 2b_1b_2 \\ &= 2b_1(b_1 + b_2) \\ &= 2b_1b. \end{aligned}$$

Substituting  $2b_1b$  for the numerator of Equation A1, we get

$$\frac{2b_1b}{2ab} = \frac{b_1}{a} = \cos(\theta).$$

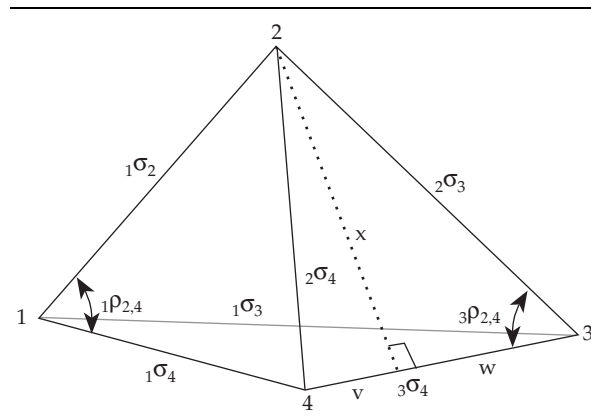
Thus, we have proven the cosine rule:

$$\cos(\theta) = \frac{a^2 + b^2 - c^2}{2ab} = \frac{b_1}{a}.$$

## Appendix 2: Risks and Correlations for More than Three Currencies

The goal is to find the correlation between Currencies 2 and 4, from the perspective of Currency 3 ( ${}_3\rho_{2,4}$ ). The process is illustrated geometrically in Figure A2.

**Figure A2. Correlations with More than Three Currencies**



From the fact that the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides, we can identify several relationships:

$$\begin{aligned} v^2 + x^2 &= 2\sigma_4^2 \Rightarrow x^2 = 2\sigma_4^2 - v^2 \\ w^2 + x^2 &= 3\sigma_2^2 \Rightarrow w^2 + 2\sigma_4^2 - v^2 = 3\sigma_2^2 \\ v + w &= 3\sigma_4 \Rightarrow v = 3\sigma_4 - w. \end{aligned}$$

Because  $v + w = 3\sigma_4$ , or  $v = 3\sigma_4 - w$ , we can substitute for  $v$ :

$$w^2 + 2\sigma_4^2 - (3\sigma_4 - w)^2 = 3\sigma_2^2.$$

By expansion,

$$w^2 - (3\sigma_4^2 - 2w3\sigma_4 + w^2) = 3\sigma_2^2.$$

Solving for  $w$  provides

$$w = \frac{3\sigma_2^2 + 3\sigma_4^2 - 2\sigma_4^2}{2({}_3\sigma_4)}.$$

From Appendix 1, we know that  ${}_3\rho_{2,4} = w/{}_3\sigma_2$ . Therefore,

$${}_3\rho_{2,4} = \frac{3\sigma_2^2 + 3\sigma_4^2 - 2\sigma_4^2}{2({}_3\sigma_4)({}_3\sigma_2)}.$$

By applying the result derived in Appendix 1 and illustrated by the computation of  ${}_{\text{DEM}}\sigma_{\text{GBP}}$  in the text,

$${}_1\rho_{2,4} = \frac{{}_1\sigma_2^2 + {}_1\sigma_4^2 - {}_2\sigma_4^2}{2({}_1\sigma_2)({}_1\sigma_4)}.$$

Therefore,

$${}_2\sigma_4 = \sqrt{{}_1\sigma_2^2 + {}_1\sigma_4^2 - 2({}_1\sigma_2)({}_1\sigma_4)({}_1\rho_{2,4})}.$$

Finally,

$${}_3\rho_{2,4} = \frac{{}_3\sigma_2^2 + {}_3\sigma_4^2 - [{}_1\sigma_2^2 + {}_1\sigma_4^2 - 2({}_1\sigma_2)({}_1\sigma_4)({}_1\rho_{2,4})]}{2({}_3\sigma_4)({}_3\sigma_2)}.$$

## Notes

1. Indeed, in the long run, EMU dictates a single currency and correlations equal to one.
2. For a general discussion of the geometric framework, see Zerolis (1996a).
3. This optimization is constrained so that only long currency positions are allowed. No net short currency positions are in the feasible opportunity set.
4. The variances of  $x$  and  $y$  are equal to  $\frac{\sum (x_i - \bar{x})^2}{n}$  and  $\frac{\sum (y_i - \bar{y})^2}{n}$ , respectively.
5. Adding a fifth currency implies moving to four dimensions, but this step is not necessary because the base-currency conversion can be done in a piecewise manner. So, three dimensions are the most that are truly necessary.
6. A covariance matrix also can be translated using linear algebra. For example, a Cholesky decomposition can be used to pivot the matrix around a selected vertex.
7. A zero correlation would correspond to the cosine of a 90 degree angle. Thus, two uncorrelated currencies would lie at a right angle to each other from the perspective of the base currency vertex. Similarly, a correlation of 1 corresponds to the cosine of zero degrees (-1 correlation equals the cosine of 180 degrees). Thus, two perfectly correlated currencies would correspond to a "collapsed" triangle—the triangle would flatten into a straight line.
8. For the more general case for  $n$  currencies, see Zerolis (1996b).

## References

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